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CONTACT ON A TRANSVERSELY ISOTROPIC HALF-SPACE, OR BETWEEN TWO TRANSVERSELY ISOTROPIC BODIES

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Abstract—There is a direct correspondence between the integral equations of contact, (that is the equations relating surface displacements to surface stresses), on a linearly elastic, homogeneous, transversely isotropic half-space and those on a similar isotropic half-space. There is also a direct correspondence between the equations for these problems, and those of contact between two transversely isotropic bodies. Thus any solution, (that is surface stress distribution), of a problem of contact, (whether frictionless, adhesive or frictional), between a rigid body and an isotropic half-space, gives a solution of the more general problem.

1. INTRODUCTION

The philosophy behind many existing solutions of problems of contact between isotropic bodies, is to find the unknown surface stresses directly from the known surface displacements and stresses by inversion of the integral equations of contact relating them. In this way we avoid the need to consider the entire stress/displacement state in either body until the problem has been reduced to a traction boundary value problem.

The equations of contact on a linearly elastic, homogeneous, isotropic half-space take very simple forms for problems with axi-symmetric or two-dimensional geometries. For instance Spence[14], has expressed them in the form

$$(1/\epsilon)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} K_{11} & -\gamma K_{12} \\ -\gamma K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{1.1}$$

The K_{ij} are integral operators and γ and ϵ elastic constants defined in the next section. Further p_1, p_2, u_1, u_2 are normal and tangential surface tractions and displacements respectively.

In the case of anisotropic materials, Dahan and Zarka[3], remark, "...the basic equations are far more complicated and few results are known". In this note it is shown that very similar results hold in the more general case, that treated by Dahan and Zarka, in which the half-space is transversely isotropic with preferred direction normal to the surface. In spite of the fact that the number of parameters required to describe the behaviour of the half-space increases from 2 to 5, only one extra parameter appears in the equations of contact. Further the K_{ij} are identical. In place of (1.1) we obtain

$$(1/\epsilon)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha K_{11} & -\gamma K_{12} \\ -\gamma K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{1.2}$$

There are many ways of obtaining the integral equations of contact on an isotropic half-space, but one of the most direct is that given by Spence[14]. He obtained the equations by integrating the Green's function, that is the oblique point-load solution, over an unknown surface stress distribution. In the next section it is shown that since the point-load solution on a transversely isotropic half-space differs only by the elastic constant multiplying the normal stress in the normal displacement equation, the equations of contact differ only in this way.

The normal-point-load solution has been obtained in the past for a completely general anisotropic material by Willis[20], and for a transversely isotropic material by Willis, Green and Zerna[8], Chapter 5, and Lekhnitski[11]. The oblique-point-load is derived in Appendix A by extending the method of Green and Zerna. The solution obtained in this paper shows a small extension of that obtained by the above authors. In their case the elastic parameters, i.e. the product $\alpha\epsilon$, are indeterminate in the isotropic limit, (Willis eqns (6.2–4), Green and Zerna

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eqn (5.12.20)), whereas the form given here reduces simply to the isotropic value. Further given the five surface parameters defined in the next section it is possible to calculate the five elastic constants of the half space (Appendix B). This is not obviously true of the form obtained by the above authors.

Equations (1.1) and (1.2), establish an isomorphism between the solutions of contact problems for materials possessing different isotropies. Another such isomorphism exists for contact between dissimilar materials linked by particular kinds of boundary conditions. It is well known that certain cases of contact between two linearly elastic isotropic bodies with different properties, are mathematically reducible to contact between one rigid and one elastic body. For example, Mindlin[12], considers tangential contact, Goodman[6] and Spence[14], adhesive normal contact, and Spence[16], frictional normal contact. By considering the oblique-point-load solution, the Green's function from which the contact equations are derived, it is shown in Section 3 that this isomorphism continues to hold for contact between any two transversely isotropic bodies, for which the preferred direction is normal to the surface, and for any geometry.

Thus any solution of a problem of contact between a rigid and an isotropic body gives a solution of the same problem of contact between two transversely isotropic bodies. This is illustrated in Section 4 by considering adhesive normal contact.

2. THE EQUATIONS OF CONTACT ON A TRANSVERSELY ISOTROPIC HALF-SPACE

Consider a system of Cartesian coordinates (x, y, z) with $\mathbf{r} = (x, y)$ and $r^2 = x^2 + y^2$. Let

$$u_x(\mathbf{r}, z),\ u_y(\mathbf{r}, z) \text{ and } u_z(\mathbf{r}, z) \tag{2.1}$$

be the components of displacement,

$$\sigma_{xx}(\mathbf{r}, z),\ \sigma_{yy}(\mathbf{r}, z),\ \sigma_{zz}(\mathbf{r}, z),\ \sigma_{xy}(\mathbf{r}, z),\ \sigma_{yz}(\mathbf{r}, z),\ \sigma_{zx}(\mathbf{r}, z) \tag{2.2}$$

be the components of stress, and $\epsilon_{xx}(\mathbf{r}, z)$, etc. be the components of strain. Consider also the half-space $z < 0$, with surface $z = 0$.

For a transversely isotropic material there are 5 independent, non-zero components of the stiffness or compliance tensor. We use the engineering components of the compliance tensor given by Lekhnitski[11]. This is expressed in terms of the compliance matrix, s_{ij}, by

$$E\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\\ 2\epsilon_{xz}\\ 2\epsilon_{yz}\\ 2\epsilon_{xy}\end{bmatrix} = \begin{bmatrix} 1 & -\nu_H & -\nu_V & & & \\ -\nu_H & 1 & -\nu_V & & & \\ -\nu_V & -\nu_V & \lambda & & & \\ & & & 2(1+\nu) & & \\ & & & & 2(1+\nu) & \\ & & & & & 2(1+\nu_H)\end{bmatrix}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xz}\\ \sigma_{yz}\\ \sigma_{xy}\end{bmatrix}. \tag{2.3}$$

The shear modulus in the $0xy$-plane, G_H, is related to E and ν_H by isotropy in that plane, $G_H = E/(2(1+\nu_H))$. By analogy the shear modulus in the $0xy$- and $0yz$-planes, G_V, has been written in terms of the parameter, ν, $G_V = E/(2(1+\nu))$.

It is shown in Appendix A that if a point force $\mathbf{F} = (Q_x, Q_y, P)$ acts at the origin on the surface of a transversely isotropic half-space the resulting surface displacements are

$$(1/\epsilon)\begin{bmatrix}u_x(\mathbf{r},0)\\ u_y(\mathbf{r},0)\\ u_z(\mathbf{r},0)\end{bmatrix} = (1/2\pi r)\begin{bmatrix}\gamma(x/r)P + Q_x + \delta(x/r)T\\ \gamma(y/r)P + Q_y + \delta(y/r)T\\ \alpha P - \gamma T\end{bmatrix}\dagger \tag{2.4}$$

†The author in collaboration with D. A. Spence has since obtained the form of eqn (2.4), with four unknown constants, α, γ, δ, ϵ, directly from similarity in the plane $z = 0$. The material in $z < 0$ was required only to be isotropic in the $0xy$-plane, and to obey Betti's reciprocal theorem and the principle of superposition. This did not give the form of the elastic constants, α, etc. It is hoped to include this derivation in a joint article.

Introduction of a length scale, as in the case of an elastic body bounded by parallel planes, destroys the similarity. We do not therefore expect the isomorphism to extend to the layer effect described by Goodman and Keer[7].

where $T = (xQ_x + yQ_y)/r)$ and

$$\alpha = \left(\frac{\lambda - \nu_V^2}{1 - \nu_H^2}\right)^{1/2}$$

$$\beta = \frac{(1+\nu) - \nu_V(1+\nu_H)}{(1-\nu_H^2)},$$

$$\gamma = \left(\frac{2}{\alpha+\beta}\right)^{1/2}\left(\frac{\alpha}{2} - \frac{\nu_V}{2(1-\nu_H)}\right), \tag{2.5}$$

$$\delta = \left(\left(\frac{2}{\alpha+\beta}\right)^{1/2}\left(\frac{1+\nu}{1+\nu_H}\right)^{1/2}\left(\frac{1}{1-\nu_H}\right) - 1\right),$$

$$\epsilon = \left(\frac{\alpha+\beta}{2}\right)^{1/2}\left(\frac{1-\nu_H}{G_H}\right).$$

For an isotropic material $\lambda = 1$ and $\nu = \nu_H = \nu_V$. Thus $\alpha = \beta = 1$, $\gamma = (1-2\nu)/(2-2\nu)$, $\delta = \nu/(1-\nu)$ and $\epsilon = (1-\nu)/G$. The point load solution for an isotropic half-space was obtained with these values of the elastic constants by Landau and Lifshitz[10] eqn (8.19).

Spence obtains the integral equations of contact in the following way. Equation (2.4) has the form

$$(1/\epsilon)\mathbf{u}(\mathbf{r}) = G(\mathbf{r}-\mathbf{s})\mathbf{F}(\mathbf{s}), \tag{2.6}$$

so that the surface displacement corresponding to a surface stress distribution

$$\boldsymbol{\sigma}(\mathbf{r}) = (\sigma_{xz}(\mathbf{r},0),\ \sigma_{yz}(\mathbf{r},0),\ \sigma_{zz}(\mathbf{r},0))$$

is

$$(1/\epsilon)\mathbf{u}(\mathbf{r}) = \int_{S_0} G(\mathbf{r}-\mathbf{s})\boldsymbol{\sigma}(\mathbf{s})\,\mathrm{d}S. \tag{2.7}$$

The region S_0 is the region of surface over which the stress $\boldsymbol{\sigma}$ is non-zero. For axi-symmetric stress distributions we assume that surface stresses, $p(r) = -\sigma_{zz}(r,0)$, $q(r) = \sigma_{rz}(r,0)$, are non-zero on a circle of contact $r \le a$. They are then related to the surface displacements, $w(r) = -u_z(r,0)$, $u(r) = u_r(r,0)$ through the integral equations of contact,

$$\begin{aligned}(1/\epsilon)w(r) &= \alpha\int_0^a k_{11}(r,s)p(s)s\,\mathrm{d}s - \gamma\int_0^a k_{12}(r,s)q(s)s\,\mathrm{d}s,\\ (1/\epsilon)u(r) &= -\gamma\int_0^a k_{21}(r,s)p(s)s\,\mathrm{d}s + \int_0^a k_{22}(r,s)q(s)s\,\mathrm{d}s.\end{aligned} \tag{2.8}$$

The kernels are given in terms of Bessel functions of the first kind by

$$k_{ij}(r,s) = \int_0^\infty J_{i-1}(\rho r)J_{j-1}(\rho s)\,\mathrm{d}\rho.$$

These equations were previously derived by Nobel and Spence[13], in the isotropic case.

Likewise for two-dimensional stress distributions we assume that the surface stresses, $p(x) = -\sigma_{zz}(x,0)$, $q(x) = \sigma_{xz}(x,0)$, are non-zero over a strip of contact, $|x| < a$, and find that they are related to the surface displacements, $w(x) = -u_z(x,0)$, $u(x) = u_x(x,0)$ through the integral equations

$$\begin{aligned}(1/\epsilon)w'(x) &= \frac{\alpha}{\pi}\int_{-a}^a \frac{p(t)\,\mathrm{d}t}{t-x} - \gamma q(x),\\ (1/\epsilon)u'(x) &= +\gamma p(x) + \frac{1}{\pi}\int_{-a}^a \frac{q(t)\,\mathrm{d}t}{t-x}.\end{aligned} \tag{2.9}$$

These were previously derived by Galin[5], and in integrated form by Timoshenko and Goodier[18], in the isotropic case.

If we make the scaling

$$p \to \alpha^{-1/2} p, \; w \to \alpha^{1/2} w, \; \gamma \to \alpha^{1/2} \gamma \tag{2.10}$$

eqns (2.8), (2.9) reduce immediately to their isotropic form. Thus the solution to a problem whose boundary conditions involve linear equality or inequality conditions, including many frictionless, adhesive or frictional contact problems, can be obtained from the corresponding isotropic problem by a simple change in the coefficients.

Finally, many limiting solutions have been obtained on almost incompressible isotropic half-spaces (see, e.g. Spence[15, 16], Turner[19]). For an incompressible material $\nu = \frac{1}{2}$, or $\gamma = 0$. If we can neglect the second term on the RHS of the first equation in (2.8) or (2.9) we obtain the normal stress directly as the frictionless solution, Hertz[9]. If it can be further demonstrated that $u(r)$ or $u'(x)$ is $O(\gamma)$ then we find from the second of (2.8) or (2.9) that the shear stress is $O(\gamma)$. Thus the terms neglected are $O(\gamma^2)$. For most real materials, $0.25 < \nu < 0.5$, γ^2 is less than 0.1. For an incompressible transversely isotropic material $\lambda = 2\nu_V$, and $\nu_V = 1 - \nu_H$. Hence $\alpha = 1$ and $\gamma = 0$, the same important result holding.

3. CONTACT BETWEEN TWO ELASTIC BODIES

We consider the problem of frictional contact between two elastic bodies. They are pressed together so that they come in contact over a region, S_0, of their common surface (the smallest typical dimension of which is denoted by a and assumed to be $O(1)$). If the total relative approach of the two bodies is w_0, and is such that contact stress is $O(1)$ then w_0 is $O(\epsilon)$ where ϵ is defined by (2.5). If terms of $O(\epsilon)$ can be ignored against terms of $O(1)$ then the two bodies may be treated as linearly elastic. (Note that the linear elastic assumption requires that the coordinate system of each body remain fixed with respect to the undeformed state of that body. A relative displacement of the two bodies therefore involves a relative displacement of the two coordinate systems.) If the smallest radius of curvature of the two bodies in the contact region is R, such that the ratio (a/R) is $O(\epsilon)$, then the equations may be applied to both bodies as if they are half-spaces. This involves an approximation of $O(\epsilon^{1/2})$. We now make this (common) half-space approximation. (Note that this approximation is still valid if one of the bodies is rigid and contains a finite number of corners in the contact region.)

Since the equations are linear the stress transmitted between the two half-spaces does not depend on the absolute surface displacements of either body, but only on the relative displacement of the two. Thus the problem of contact between two elastic bodies can be transformed into that of contact between a rigid body and an elastic half-space. The elastic properties of this half-space can be obtained as a combination of those of the original bodies, and its surface displacements are the relative surface displacements. The transformation has been deduced in the past from the isotropic form of (2.8) or (2.9) (see, e.g. Spence[17]). However the transformation is valid for contact between two transversely isotropic bodies. This is shown in this section by deriving it directly from the point-load solution. The corresponding integral equations of contact can then be obtained by integrating the resultant over the contact region S_0 (eqn 2.7).

There is a further advantage in deriving the transformation directly from the point-load. It is also the Green's function for the problem considered by England[4], and Clements[1, 2], namely a crack between two half-spaces bounded over their entire common surface, $z = 0$, except in the region of the crack. Thus the surface stress distributions found by Clements for transversely isotropic half-spaces could be deduced directly from those found by England for isotropic half-spaces, and these could be calculated by considering a crack between an isotropic and a rigid half-space.

The transformation is derived in the following way. Consider two half-spaces, $i = 1, 2$, with elastic parameters α^i, β^i, γ^i, δ^i and ϵ^i. The surface displacement of the half-space, i, resulting from a point load $\mathbf{F}^i = (Q_x^i, Q_y^i, P^i)$ acting at the origin is, from eqn (2.5),

$$\begin{bmatrix} u_x^i(\mathbf{r},0) \\ u_y^i(\mathbf{r},0) \\ u_z^i(\mathbf{r},0) \end{bmatrix} = \frac{\epsilon^i}{2\pi r} \begin{bmatrix} \gamma^i(x/r)P^i + Q_x^i + \delta^i(x/r)T^i \\ \gamma^i(y/r)P^i + Q_y^i + \delta^i(y/r)T^i \\ \alpha^i P^i \quad - \gamma^i \quad T^i \end{bmatrix}, \tag{3.1}$$

where $T^i = (xQ_x^i + yQ_y^i)/r)$. The normal is assumed positive out of each half-space. The half-space $i = 1$ has a right-hand system of Cartesian, but $i = 2$ a left-hand system. Thus the x- and y-axis correspond but the z-axes are anti-parallel. If a point-load is transferred between these two half-spaces static equilibrium requires that

$$P = P^1 = P^2,\ Q_x = Q_x^1 = -Q_x^2, Q_y = Q_y^1 = -Q_y^2. \tag{3.2}$$

The relative surface displacement, with respect to the coordinates of body, $i = 1$, are

$$\begin{aligned} &u_x(\mathbf{r},0) = u_x^1(\mathbf{r},0) - u_x^2(\mathbf{r},0),\ u_y(\mathbf{r},0) = u_y^1(\mathbf{r},0) - u_y^2(\mathbf{r},0), \\ &u_z(\mathbf{r},0) = u_z^1(\mathbf{r},0) + u_z^2(\mathbf{r},0). \end{aligned} \tag{3.3}$$

By subtracting the first and second of each of eqns (3.1) and adding the third we find that the relative surface displacements are related to the transferred point-load, $F = (Q_x, Q_y, P)$, through eqn (2.4) if

$$\begin{aligned} &\epsilon = \epsilon^1 + \epsilon^2, \qquad \alpha\epsilon = \alpha^1\epsilon^1 + \alpha^2\epsilon^2, \\ &\gamma\epsilon = \gamma^1\epsilon^1 - \gamma^2\epsilon^2,\ \delta\epsilon = \delta^1\epsilon^1 + \delta^2\epsilon^2. \end{aligned} \tag{3.4}$$

(The fifth coefficient β of course does not appear.) This is the transformation used extensively in the literature for contact between isotropic bodies with axi-symmetric and two dimensional geometries. In that case we obtain only the first and third of eqns (3.4). It was effectively used first by Hertz[9], to show that normal contact between like materials is frictionless, $\gamma = 0$.

Two points are noted.

(i) The problem of contact between two transversely isotropic half-spaces transforms into one of contact between a rigid and an effective transversely isotropic half-space. The value of β remains to be chosen at will. It is shown in Appendix B that the elastic coefficient, E, λ, ν, ν_H and ν_V of the effective half-space can be calculated from the coefficients α, β, γ, δ and ϵ.

(ii) If both materials are isotropic $\alpha = \alpha^1 = \alpha^2 = 1$, but in general γ and δ will not correspond to an effective isotropic material. However δ does not enter into the axi-symmetric or two-dimensional equations so for these geometries an equivalent isotropic half-space can be found.

4. EXAMPLE

It has been shown that the problem of contact between two transversely isotropic, linearly elastic bodies for which the preferred direction is normal to the surface can be transformed into an equivalent problem of a rigid body in contact with a transversely isotropic half-space, the elastic properties of which can be obtained in terms of those of the original two bodies. It was further shown that because the equations of contact on a transversely isotropic half-space differ from those on an isotropic half-space only by the elastic constant multiplying the normal stress in the normal displacement equation, the problem can be solved by considering only contact between a rigid body and an isotropic one.

These ideas are illustrated by the solution of a simple example. That considered is the normal adhesive contact of spheres. The isotropic problem was solved by Goodman[6], and later by Spence[14]. Spence's solution to the idealized problem of a rigid sphere of radius, R, indenting an isotropic half-space is recalled. The solution for a rigid sphere and a transversely isotropic half-space follows immediately from the results of Section 2. Finally two spheres of radii, R_1, R_2 and elastic properties, α^1, α^2, etc. are considered. The solution of adhesive contact of two such spheres is obtained using eqns (3.2)–(3.4).

From similarity consideration Spence shows that if a sphere of radius R is pressed into a half-space with conditions of adhesion over the interface, the normal and tangential displacements take the following form in the contact region, $r \leqslant a$,

$$w(r) = W(1 - r^2/2R), \quad u(r) = -WAr^2. \tag{4.1}$$

The depth of penetration, W, is related to R and a, and the coefficient A is given in terms of the elastic parameters of the half-space. Both are initially unknown, but are calculated from the fact that the normal stress $p(r)$ and tangential stress $q(r)$ vanish on $r = a$.

Starting from the isotropic form of (2.8) Spence also shows that $p(r)$ and $q(r)$ obey a single Fredholm equation. Defining $f(\rho)$ by

$$f(\rho) = \bar{p}(\rho) - \bar{q}(\rho), \tag{4.2}$$

where $\bar{p}(\rho)$ and $\bar{q}(\rho)$ are the Hankel transforms

$$\bar{p}(\rho) = \int_0^a p(r)J_0(pr)r \, \mathrm{d}r, \quad \bar{q}(\rho) = \int_0^a q(r)J_1(\rho r)r \, \mathrm{d}r,$$

he shows that $f(\rho)$ satisfies the equation

$$f(\rho) + \phi_1 \int_0^\infty k(\rho - \sigma)f(\sigma) \, \mathrm{d}\sigma = \phi_2 c(\rho), \tag{4.3}$$

where

$$\begin{aligned}
\phi &= 2 - 4\nu = (2\gamma/(1-\gamma)), \\
\phi_2 &= (4G/\pi) = (2/\epsilon\pi(1-\gamma)), \\
k(\rho) &= \sin \rho/\pi\rho, \\
c(\rho) &= \int_0^a (w^*(s) \cos \rho s - u^*(s) \sin \rho s) \, \mathrm{d}s, \\
w^*(r) &= \frac{\mathrm{d}}{\mathrm{d}r} \int_0^r \frac{w(s)s \, \mathrm{d}s}{(r^2 - s^2)^{1/2}}, \quad u^*(r) = \int_0^r \frac{(su(s))' \, \mathrm{d}s}{(r^2 - s^2)^{1/2}}.
\end{aligned}$$

Spence inverts eqn (4.3) by the Weiner–Hopf technique to show that

$$W = \frac{a^2}{\theta(\kappa)R}, \quad A = \frac{4\kappa^2(\phi_1 + 1)^{1/2}}{3\phi_1} = \frac{2\kappa^2}{3\gamma}(1 - \gamma^2)^{1/2}, \tag{4.4}$$

where

$$\begin{aligned}
\kappa &= \frac{1}{\pi}\ln(1 + \phi_1) = \frac{1}{\pi}\ln\left\{\frac{1+\gamma}{1-\gamma}\right\}, \\
\theta(\kappa) &= 1 - 0.6931\kappa^2 + 0.2254\kappa^4 + \cdots.
\end{aligned}$$

He also obtains integral expressions for $p(r)$ and $q(r)$, which will not be quoted, and an expression for the total normal force P,

$$P = \int_0^a p(r)(2\pi r) \, \mathrm{d}r.$$

Table 1. Parameters for three materials, I: isotropic, $\nu = 0.2$, II; t-isotropic, $\lambda = 1.66$, $\nu_H = 0.2$, $\nu_V = 0.16$, $\nu = 0.25$, III; t-isotropic, $\lambda = 0.6$, $\nu_H = 0.2$, $\nu_V = 0.16$, $\nu = 0.25$

	I	II	III
κ	0.25097	0.27608	0.23527
θ	0.95724	0.94848	0.96233
A	0.10381	0.11358	0.09761
$(3\epsilon PR/4\pi a^3)$	0.69926	0.51812	0.86015

In fact

$$P = \frac{4\phi_2\pi^2 a^3}{3R}\frac{\kappa}{\phi_1}, \text{ or } \frac{3\epsilon PR}{4\pi a^3} = (\kappa/\gamma). \tag{4.5}$$

With isotropy in the $0xy$-plane, the same similarity arguments can be applied to a transversely isotropic material, so the displacement conditions (4.1) continue to hold. Further defining $f(\rho)$ by

$$f(\rho) = \alpha^{1/2}\bar{p}(\rho) - \bar{q}(\rho), \tag{4.6}$$

it is deduced from (2.10), that for a transversely isotropic half-space (4.3) holds, but with ϕ_1 and ϕ_2 given by

$$\phi_1 = \frac{2\gamma}{\alpha^{1/2} - \gamma}, \quad \phi_2 = \frac{2}{\epsilon\pi(\alpha^{1/2} - \gamma)}. \tag{4.7}$$

Thus Spence's solution is a solution in the transversely isotropic case with

$$A = \frac{2\kappa^2}{3\gamma}(\alpha - \gamma^2)^{1/2}$$

$$\kappa = \frac{1}{\pi}\ln\left\{\frac{\alpha^{1/2} + \gamma}{\alpha^{1/2} - \gamma}\right\}, \tag{4.8}$$

and

$$\frac{3\epsilon PR}{4\pi a^3} = (\kappa/\gamma\alpha^{1/2}).$$

Values of κ, θ, A and $(3\epsilon PR/4\pi a^3)$ are tabulated in Table 1 for 3 materials; I, an isotropic material, $\nu = 0.2$; II, a transversely isotropic material weaker in the preferred direction, $\lambda = 1.66$, $\nu_H = 0.2$, $\nu_V = 0.16$, $\nu = 0.25$, (London Clay); III, stiffer in the preferred direction, $\lambda = 0.6$, $\nu_H = 0.2$, $\nu_V = 0.16$, $\nu = 0.25$.

Using (3.2)–(3.4) it is deduced that the solution to the problem of contact between two different spheres is equivalent to the problem of contact between a rigid sphere of radius, $R = R_1R_2/(R_1 + R_2)$, and a half-space whose elastic parameters are related to those of the two spheres through (3.4). The solution is given by (4.8). Finally, if the half space is incompressible $\gamma = \kappa = 0$. However the ratio (κ/γ) is finite. In fact

$$\frac{\kappa}{\gamma} = \left\{\frac{2}{\pi\alpha^{1/2}}\right\} \tag{4.9}$$

whence

$$W = (a^2/R), \quad A = 0,$$

and

$$P = \frac{8a^3}{3\epsilon R}, \quad (\alpha \to 1).$$

These are the frictionless values, Hertz[6].

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APPENDIX A

Derivation of the oblique-point-load solution on a transversely isotopic half-space

A1 Potential function formulation. To obtain the point-load solution the constitutive equation is written in terms of the stiffness tensor, (Green and Zerna[8], eqn 5.12.1),

$$\frac{1}{E}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xz}\\ \sigma_{yz}\\ \sigma_{xy}\end{bmatrix}=\begin{bmatrix}c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{11} & c_{13} & & & \\ c_{13} & c_{13} & c_{33} & & & \\ & & & c_{44} & & \\ & & & & c_{44} & \\ & & & & & \frac{1}{2}(c_{11}-c_{12})\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\\ 2\epsilon_{xz}\\ 2\epsilon_{yz}\\ 2\epsilon_{xy}\end{bmatrix} \tag{A1}$$

The components of the stiffness tensor can be found in terms of those of the compliance tensor by inversion of (2.3).

Defining now the Cartesian displacement vector as (u, v, w), we obtain the equations relating the displacement of a transversely isotropic body, given by Green and Zerna, (5.12.3-5),

$$\left[\tfrac{1}{2}(c_{11}-c_{12})\nabla_H^2+c_{44}\frac{\partial^2}{\partial z^2}\right]u+\frac{\partial}{\partial x}\left[\tfrac{1}{2}(c_{11}+c_{12})\Delta_H+(c_{13}+c_{44})\frac{\partial w}{\partial z}\right]=0, \tag{A2}$$

$$\left[\tfrac{1}{2}(c_{11}-c_{12})\nabla_H^2+c_{44}\frac{\partial^2}{\partial z^2}\right]v+\frac{\partial}{\partial y}\left[\tfrac{1}{2}(c_{11}+c_{12})\Delta_H+(c_{13}+c_{44})\frac{\partial w}{\partial z}\right]=0, \tag{A3}$$

$$\left[c_{44}\nabla_H^2+c_{33}\frac{\partial^2}{\partial z^2}\right]w+\frac{\partial}{\partial z}\left[(c_{13}+c_{44})\Delta_H\right]=0, \tag{A4}$$

where

$$\nabla_H^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\text{ and }\Delta_H=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}.$$

Green and Zerna derive a potential function formulation of these equations. They show that a general solution to (A2)–(A4) is given by

$$u=\frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_2}{\partial x}+\frac{\partial\phi_3}{\partial y}, \tag{A5}$$

$$v=\frac{\partial\phi_1}{\partial y}+\frac{\partial\phi_2}{\partial y}-\frac{\partial\phi_3}{\partial x}, \tag{A6}$$

$$w=k_1\frac{\partial\phi_1}{\partial z}+k_2\frac{\partial\phi_2}{\partial z}, \tag{A7}$$

where

$$\left(\nabla_H^2 + \nu_i \frac{\partial^2}{\partial z^2}\right)\phi_i = 0, i = 1, 2, 3, \text{(no sum implied)}. \quad \text{(A8)}$$

In eqns (A5)–(A8), v_1 and v_2 are the roots of the quadratic equation

$$\alpha^2 \nu^2 - 2\beta\nu + 1 = 0, \quad \text{(A9)}$$

where

$$\alpha^2 = (c_{11}/c_{33}),$$

$$\beta = \frac{c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44}}{2c_{44}c_{33}},$$

and ν_3 is given by

$$\nu_3 = \frac{2c_{44}}{c_{11} - c_{12}}.$$

After inversion of (2.3) it is possible to show that α and β are as defined in (2.5). Finally k_1 and k_2 are the roots of the quadratic equation

$$k^2 + \left[2 - \frac{c_{11}c_{33} - c_{13}^2}{c_{44}(c_{13} + c_{44})}\right]k + 1 = 0, \quad \text{(A10)}$$

and are related to ν_1 and ν_2 by

$$\frac{c_{11}\nu_i - c_{44}}{c_{13} + c_{44}} = \frac{(c_{13} + c_{44})\nu_i}{c_{33} - c_{44}\nu_i} = k_i, \quad \text{(A11)}$$

or

$$\frac{k_i(c_{13} + c_{44}) + c_{44}}{c_{11}} = \frac{k_i c_{33}}{(c_{13} + c_{44}) + k_i c_{44}} = \nu_i. \quad \text{(A12)}$$

Two points are noted.

(a) The functions ϕ_i, $i = 1, 2, 3$ are not true potential functions, but obey the equation

$$\nabla_i^2\phi_i = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_i^2}\right)\phi_i = 0, i = 1, 2, 3 \text{ no sum}, \quad \text{(A13)}$$

where $z_i = z/\mu_i$ and $\mu_i = \surd(\nu_i)$ with positive real part. Solving (A9) gives

$$\mu_1 = (1/\alpha)\surd(\beta + \surd(\beta^2 - \alpha^2)),\ \mu_2 = (1/\alpha)\surd(\beta - \surd(\beta^2 - \alpha^2)), \quad \text{(A14)}$$

whence

$$\mu_1\mu_2 = (1/\alpha) \text{ and } \mu_1 + \mu_2 = (1/\alpha)\surd 2(\beta + \alpha).$$

(b) The parameters ν_1, ν_2, k_1 and k_2 obey the following relations

$$(1 + k_1)(1 + k_2) = \frac{c_{11}c_{33} - c_{13}^2}{c_{44}(c_{13} + c_{44})}, \quad \text{(A15)}$$

$$k_1 - k_2 = \frac{c_{11}(\mu_1^2 - \mu_2^2)}{c_{13} + c_{44}}, \quad \text{(A16)}$$

$$\mu_2 k_1 - \mu_1 k_2 = \frac{(\mu_1 - \mu_2)(\mu_1\mu_2 c_{11} + c_{44})}{c_{13} + c_{44}}. \quad \text{(A17)}$$

Equation (A15) follows from (A10), (A16) is obtained by subtracting the two eqns (A11), and (A17) by multiplying by μ_2 and μ_1, respectively, and then subtracting.

The six corresponding components of the stress tensor are given by Green and Zerna (5.12.15-16). The three of interest are σ_{zz}, σ_{xz} and σ_{yz}. Using eqn (A12) we obtain

$$\sigma_{zz} = Ec_{44}\left[\nu_1(1 + k_1)\frac{\partial^2\phi_1}{\partial z^2} + \nu_2(1 + k_2)\frac{\partial^2\phi_2}{\partial z^2}\right], \quad \text{(A18)}$$

$$\sigma_{xz} = Ec_{44}\left[(1 + k_1)\frac{\partial^2\phi_1}{\partial x\partial z} + (1 + k_2)\frac{\partial^2\phi_2}{\partial x\partial z} + \frac{\partial^2\phi_3}{\partial y\partial z}\right], \quad \text{(A19)}$$

$$\sigma_{yz} = Ec_{44}\left[(1 + k_1)\frac{\partial^2\phi_1}{\partial y\partial z} + (1 + k_2)\frac{\partial^2\phi_2}{\partial y\partial z} - \frac{\partial^2\phi_3}{\partial x\partial z}\right]. \quad \text{(A20)}$$

We are now in a position to obtain the displacements corresponding to point normal and tangential loads acting at the origin.

A2 The Boussinesq solution. If $\Phi_1(x, y, z)$ is a potential function satisfying

$$\nabla^2\Phi_1 = \frac{\partial^2\Phi_1}{\partial x^2} + \frac{\partial^2\Phi_1}{\partial y^2} + \frac{\partial^2\Phi_1}{\partial z^2} = 0,$$

then putting

$$\phi_1 = \frac{1}{\kappa}\left(\frac{\mu_1}{1+k_1}\right)\Phi_1(x, y, z_1),\ \phi_2 = \frac{1}{\kappa}\left(\frac{\mu_2}{1+k_2}\right)\Phi_1(x, y, z_2),\ \phi_3 = 0, \tag{A21}$$

where

$$\kappa = Ec_{44}(\mu_1 - \mu_2),$$

gives on the surface $z = 0$, $\sigma_{xz}(\mathbf{r}, 0) = \sigma_{yz}(\mathbf{r}, 0) = 0$, and

$$\sigma_{zz}(\mathbf{r}, 0) = \frac{\partial^2\Phi_1}{\partial z^2}(\mathbf{r}, 0). \tag{A22}$$

This is the Boussinesq solution. Using (A15)–(A17) and the inverse of (2.3) we show that

$$w(\mathbf{r}, 0) = \alpha\epsilon\frac{\partial\Phi}{\partial z}1(\mathbf{r}, 0), \tag{A23}$$

$$u(\mathbf{r}, 0) = \gamma\epsilon\frac{\partial\Phi}{\partial x}1(\mathbf{r}, 0), \tag{A24}$$

$$v(\mathbf{r}, 0) = \gamma\epsilon\frac{\partial\Phi}{\partial y}1(\mathbf{r}, 0). \tag{A25}$$

The potential corresponding to a point force P acting at the origin on the half-space $z \leq 0$ is

$$\Phi_1(x, y, z) = -\frac{P}{2\pi}\ln(R - z), \tag{A26}$$

where $R = \sqrt{(r^2 + z^2)}$, $r^2 = x^2 + y^2$. The surface displacements are the corresponding terms of (2.4).

A3 The Cerruti solution. If $\Phi_2(x, y, z)$ is a function satisfying $\nabla^2\Phi_2 = 0$, then putting

$$\phi_1 = \frac{1}{\alpha\kappa}\left(\frac{1}{1+k_1}\right)\frac{\partial\Phi_2}{\partial x}(x, y, z_1),\ \phi_2 = \frac{1}{\alpha\kappa}\left(\frac{1}{1+k_2}\right)\frac{\partial\Phi_2}{\partial x}(x, y, z_2),$$

$$\phi_3 = -\frac{\mu_3}{Ec_{44}}\frac{\partial\Phi_2}{\partial y}(x, y, z_3) \tag{A27}$$

where $\mu_3 = \sqrt{(\nu_3)}$ and $z_3 = z/\mu_3$, gives on the plane $z = 0$, $\sigma_{zz}(\mathbf{r}, 0) = \sigma_{yz}(\mathbf{r}, 0) = 0$ and

$$\sigma_{xz}(\mathbf{r}, 0) = \frac{\partial^3\Phi_2}{\partial z^3}(\mathbf{r}, 0). \tag{A28}$$

This is the Cerruti solution. The surface displacements are

$$w(\mathbf{r}, 0) = \epsilon\gamma\frac{\partial^2\Phi_2}{\partial x\partial z}, \tag{A29}$$

$$u(\mathbf{r}, 0) = \epsilon\left[\frac{\partial^2\Phi_2}{\partial z^2} - \delta\frac{\partial^2\Phi_2}{\partial y^2}\right],$$

$$v(\mathbf{r}, 0) = -\epsilon\delta\frac{\partial^2\Phi_2}{\partial x\partial y}. \tag{A30}$$

The potential corresponding to a point tangential load is

$$\Phi_2 = \frac{Q_x}{2\pi}(z\ln(R - z) + R), \tag{A31}$$

the surface displacements being the corresponding terms in (2.4).

The general solution is completed by putting

$$\phi_1 = \frac{1}{\alpha\kappa}\left(\frac{1}{1+k_1}\right)\frac{\partial\Phi_3}{\partial y}(x, y, z_1);\ \phi_2 = \frac{1}{\alpha\kappa}\left(\frac{1}{1+k_2}\right)\frac{\partial\Phi_3}{\partial y}(x, y, z_2);$$

$$\phi_3 = \frac{\mu_3}{Ec_{44}}\frac{\partial\Phi_3}{\partial x}(x, y, z_3); \tag{A32}$$

with $\Phi_3 = -(Q_y/2\pi)(z\ln(R + z) + R)$ corresponding to the point load in the y-direction.

APPENDIX B

The components of the compliance tensor

Given observed or calculated values of the elastic parameters α, β, γ, δ and ϵ, the components of the compliance tensor E, λ, ν_H, ν_V and ν can be calculated in the following way:

$$\begin{aligned}
(1-\nu_H) &= \frac{\beta_1}{(\delta+1)^2}\left(\frac{2}{\alpha+\beta}\right),\\
\left(\frac{1+\nu}{1-\nu_H{}^2}\right) &= \beta_1 = \beta+\gamma_1,\\
\left(\frac{\nu_V}{1-\nu_H}\right) &= \gamma_1 = \alpha - 2\gamma\left(\frac{\alpha+\beta}{2}\right)^{1/2},\\
\lambda &= \alpha^2(1-\nu_H{}^2)+\nu_V{}^2,\\
E &= \left(\frac{\alpha+\beta}{2}\right)^{1/2}\frac{2(1-\nu_H{}^2)}{\epsilon}.
\end{aligned} \tag{B1}$$

The components of the stiffness tensor can then be calculated by inversion of (2.3).